

## Entropy and 'The Arrow of Time': A Love Story

by

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**Abstract:** In this short paper we consider a new perspective on entropy and derive an interesting relationship between entropy and time. In light of this relationship, we show that The Second Law of Thermodynamics acquires a new meaning as stating that every physical process requires a lapse of time. In simple language, the Second Law says that 'everything takes time to happen'. Nothing happens at an instant. This defines 'the arrow of time'. Furthermore, we show that Planck's Law and Boltzmann's Entropy Equation are mathematically equivalent, that each can be derived from the other. We also obtain an equation for the number of microstates of a system, which relates to the energy of the system. The Fundamental Thermodynamic Relation likewise acquires a new and intuitive interpretation and provides further evidence for the existence of 'accumulation of energy before manifestation of energy'.

**Introduction:** While traditionally entropy has been thought of as a 'measure of disorder' and The Second Law of Thermodynamics as declaring the ultimate heat death of the Universe, our results here show such understanding of entropy and the Second Law to be unnecessary and misleading. The relationship most relevant is that between entropy and time, not entropy and disorder. The Second Law then establishes in a formal way what is clear and obvious: that any physical process takes time.

*Notation:*

$$\Delta E = E(t) - E(s)$$

$$\Delta t = \tau = t - s$$

$$E_{av} = \bar{E} = \frac{1}{t - s} \int_s^t E(u) du$$

$$\eta = P = \int_s^t E(u) du$$

In a previous paper (['Planck-like' Characterization of Exponential Functions](#)) we have proven the following mathematical characterizations. In the same referenced paper we showed that if  $E(t)$  is any integrable function the same characterizations hold, but as limits. By assuming exponential functions we get exact equations and avoid limit approximations. We have from this referenced paper the following mathematical results,

$$E(t) = E_0 e^{\nu t} \text{ if and only if } E(s) = \frac{\eta \nu}{e^{\eta \nu / E_{av}} - 1} \quad (1)$$

$$\text{For any integrable function } E(t), \lim_{t \rightarrow s} \frac{\eta \nu}{e^{\eta \nu / E_{av}} - 1} = E(s) \quad (2)$$

$$E(t) = E_0 e^{\nu t} \text{ if and only if } \Delta E = \eta \nu \quad (3)$$

$$E(t) = E_0 e^{rt} \text{ if and only if } \frac{\Delta E}{E_{av}} = r \Delta t \quad (4)$$

In the context of Physics,  $E(t)$  can be thought of as the energy of a system at time  $t$ .

**Thermodynamic Entropy and Planck's Law:** In [\*The Interaction of Measurement\*](#) we have shown that

$$E_0 = \frac{\Delta E}{e^{\Delta E/E_{av}} - 1} = \frac{\eta v}{e^{\eta v / \kappa \mathcal{T}} - 1} \quad (\text{if } E(t) \text{ is any exponential function}) \quad (5)$$

or,

$$E_0 \approx \frac{\Delta E}{e^{\Delta E/E_{av}} - 1} \approx \frac{\eta v}{e^{\eta v / \kappa \mathcal{T}} - 1} \quad (\text{if } E(t) \text{ is any integrable function}) \quad (6)$$

where,

$$\eta = \int_0^{\tau} E(t) dt \text{ is the 'accumulation of } E \text{ ' over a time interval } \tau$$

$$E_{av} = \frac{\eta}{\tau} \text{ is the 'average value of } E \text{ ' over a time interval } \tau$$

$$\Delta E = E(\tau) - E(0) \text{ is the incremental 'change of } E \text{ ' over a time interval } \tau$$

and we define  $\mathcal{T} = \left(\frac{1}{\kappa}\right) \frac{\eta}{\tau}$  where  $\kappa$  is a scalar constant. Then,  $E_{av} = \kappa \mathcal{T}$

To avoid limits and approximations, using (5) above we have the following mathematical derivations.

We have that

$$e^{\Delta E/E_{av}} = 1 + \frac{\Delta E}{E_0} = \frac{E}{E_0} \text{ and so, } \frac{\Delta E}{E_{av}} = \ln\left(\frac{E}{E_0}\right). \quad (7)$$

Letting

$$\Delta S_x = \frac{\Delta E}{\mathcal{T}} = \kappa \frac{\Delta E}{E_{av}} \quad (8)$$

we get

$$\Delta S_x = \kappa \ln\left(\frac{E}{E_0}\right) \quad (9)$$

Also, since  $E = E_0 e^{v \Delta t}$ , this becomes

$$\Delta S_x = \kappa v \Delta t \quad (10)$$

Note further that the reverse is also possible. We can start with equation (9) and derive equation (5). These are purely mathematical results independent of Physics. Consider, however, the following striking similarities to well established laws in Physics.

*Planck's Law* in Physics, written as  $E_0 = \frac{h\nu}{e^{h\nu/kT} - 1}$ , has the same exact form as (5) above. Note that

$E_{av} = kT$  (in degrees Kelvin) exactly corresponds to  $E_{av} = \kappa \mathcal{T}$  above. While Thermodynamic Entropy

$\Delta S_\theta$  can be defined as  $\Delta S_\theta = \frac{\Delta E}{T}$ . Starting with *Planck's Law*, written as  $E_0 = \frac{\Delta E}{e^{\Delta E/E_{av}} - 1}$  and using the exact same mathematical manipulations above, equation (9) above then becomes the following,

$$\Delta S_\theta = k \cdot \ln \left( \frac{E}{E_0} \right), \text{ where } k \text{ is Boltzmann's constant} \quad (11)$$

We also know from Statistical Thermodynamics *Boltzmann's Entropy Equation*,

$$S_\theta = k \ln \Omega, \text{ where } \Omega \text{ is the number of microstates} \quad (12)$$

and the change of entropy of a system is given by

$$\Delta S_\theta = k \ln \left( \frac{\Omega}{\Omega_0} \right) \quad (13)$$

Comparing (11) and (13), we have that,

$$\frac{\Omega}{\Omega_0} = \frac{E}{E_0} \quad (14)$$

From (14) we have that  $\Omega$  and  $E$  are proportional. This certainly makes physical sense, since  $\Omega$  is the number of equally likely microstates which relate directly to the energy of the system, by the equipartition principle. From (5) and (6) we know that if (11) were an *exact equation* (rather than a limit), then it must mathematically be true that  $E(t) = E_0 e^{v(t-t_0)}$ . We have then that

$$\Delta S_\theta = k \cdot \ln \left( \frac{E}{E_0} \right) = k \cdot \ln \left( e^{v \Delta t} \right) = kv \Delta t \quad (15)$$

and from (14) we have that,

$$\Omega = \Omega_0 e^{v \Delta t} \quad (16)$$

Equations (16) and (14) suggest that  $E(t) = a\Omega(t)$ . That energy is the measure of the number of microstates of a system. The above derivations suggest that *Planck's Law* and *Boltzmann's Entropy Equation* are equivalent. Each can be mathematically derived from the other, each implies the other!

**Entropy and Time:** Thermodynamic entropy  $\Delta S_\theta$  is typically defined as  $\Delta S_\theta = \Delta E/T$ , where  $\Delta E$  is energy and  $T$  is Kelvin temperature of the system. We also have that the average energy of the system is given by  $E_{av} = kT$  where  $k$  is Boltzmann's constant. In the context of Physics where  $E(t)$  is the energy of a system at time  $t$ , the quantity  $\Delta E/E_{av}$  in (4) above essentially *is entropy* up to the scalar constant  $k$ . Following through with this comparison and using (4) and (3) above we have,  $\Delta E/E_{av} = \eta r/E_{av} = r \Delta t$ . This reduces to  $\eta/E_{av} = \Delta t$ . But this is *always* true for *any* function  $E(t)$  by definition of  $E_{av}$ . This suggests the following definition of entropy:

*Definition of Entropy: The entropy  $\Delta S_\eta$  of a system with energy given by  $E(t)$  at any time  $t$  is the ratio of 'accumulation of energy'  $\eta$  over 'average energy'  $E_{av}$ , i. e.  $\Delta S_\eta = \eta/E_{av}$ .*

From the above discussion (and from (15) ) we have the following interesting relationship between entropy and time.

**Basic 'Entropy vs Time' Relationship:** a)  $\Delta S_\eta = \eta/E_{av} = \Delta t$ .  
 b)  $\Delta S_\theta = kV \Delta t$  (17)

We also have the following property of entropy,  $\Delta S_\eta$  :

*Additive Property of Entropy: If a system goes from state A to state B to state C, then the change in entropy from A to B plus the change in entropy from B to C would equal to the change in entropy from A to C.*

*Proof: Let  $\Delta t_{AB}$ ,  $\Delta t_{BC}$  and  $\Delta t_{AC}$  be the times going from A to B, B to C, and A to C respectively. Clearly  $\Delta t_{AB} + \Delta t_{BC} = \Delta t_{AC}$ . Thus  $\Delta S_{AB} + \Delta S_{BC} = \Delta S_{AC}$ , from the above Basic Relationship.*

It is noteworthy that once again the quantity 'accumulation of energy'  $\eta$  naturally comes up as more 'primary' and in terms of which entropy and many other physical quantities can be defined. (see: [Prime 'physis' and the Mathematical Derivation of Basic Law](#)).

**The Second Law of Thermodynamics:** The Second Law simply states that in any physical process the change in entropy is positive, i.e. ,  $\Delta S > 0$  . If we were to understand entropy as giving us a measure of randomness or disorder, the Second Law would then be interpreted to mean that the Universe is going from a more orderly state to a more random and chaotic state, and ultimately death.

The above result alters such misleading interpretation of both entropy and the Second Law. Rather, from the above *Basic Relationship* (17) we have that when  $\Delta S_\eta > 0$ , or  $\Delta S_\theta > 0$  then also  $\Delta t > 0$ , and visa-versa. The Second Law of Thermodynamics would then be interpreted to mean that there must be a positive lapse of time with every physical process. This makes intuitive sense and is self evident. The Second Law of Thermodynamics then simply specifies  $\Delta t$  as being positive. According to our *Basic Relationship*, Entropy and Time are intimately embraced to create 'the arrow of time'.

**The Fundamental Thermodynamic Relation:** It is a well known fact that the internal energy  $U$ , entropy  $S$ , temperature  $T$ , pressure  $P$  and volume  $V$  of a system are related by the following equation.

$$dU = TdS - PdV \quad (18)$$

By using increments rather than differentials, and using the fact that work performed by the system is given by  $W = \int PdV$ , equation (18) can be re-written as,

$$\Delta S = \frac{\Delta U}{T} + \frac{\Delta W}{T} \quad (19)$$

All the terms in equation (19) can be thought as being various entropy quantities. Note then that the *fundamental thermodynamic relation* can be interpreted as saying that,

*“the change of entropy of a system equals the sum of the change in the internal (unmanifested) plus the change in the external (manifested) entropy of the system”*

In light of the above discussion, and using our *Basic Relationship* (17), this *fundamental thermodynamic relation* would then say that,

*“the lapsed time for a physical process equals the time for the 'accumulation of energy' plus the time for the 'manifestation of energy' for the process to occur”*

This interpretation of the *fundamental thermodynamic relation* also has merit even if we were to use the more standard definition of thermodynamic entropy, in light of our result (15) above.

#### **Notes:**

- The definition of entropy  $\Delta S_\eta$  given above differs from the thermodynamic entropy  $\Delta S_\theta$  by a constant determined by the system. More exactly, we have  $\Delta S_\theta = rk\Delta S_\eta$ , where  $k$  is Boltzmann's constant and  $r$  is a growth factor determined by the system. *Basic Relationship* (17) above then becomes  $\Delta S_\theta = rk \Delta t$ . The simplicity of the above definition of entropy  $\Delta S_\eta$ , along with the resulting *Relationship* (17), are compelling. This understanding of entropy comes closer to the views of entropy as 'dispersal of energy' or 'spreading of energy' that have been gaining favor with some recently. ([reference](#))
- Often entropy can be thought as the measure of 'available energy' to do work. This is perfectly in harmony with our results above. Since entropy  $\Delta S_\eta$  is related to the time  $\Delta t$  to transition from one state to another, the longer that transition takes, the less work is possible. Thus, the higher the entropy  $\Delta S_\eta$ , the less 'available energy' there is to do work.
- The definition of entropy given above and the relationship between entropy and time established clearly shows the reciprocal relationship between entropy and temperature. The higher the entropy, the higher the transition time of the system, the lower the temperature of the system. It all fits harmoniously well together.

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