A 'Planck-like' Characterization of Exponential Functions

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Abstract. We derive a characterization of simple exponential functions that has the exact mathematical form to Planck's Formula for blackbody radiation in Quantum Physics.

Notation:

$$\Delta E = E(t) - E(s)$$
$$\Delta t = \tau = t - s$$
$$E_{av} = \bar{E} = \frac{1}{t-s}\int_s^t E(u)du$$
$$\eta = P = \int_s^t E(u)du$$

Note: All proofs can be found in section ***3. Mathematical Derivations*** *of this note.*

1. Main Derivations

The central mathematical result is the following characterization of simple exponential functions:

Characterization 2a: $E(t) = E_0 e^{\nu t}$ if and only if $E(s) = \dfrac{\eta\nu}{e^{\eta\nu/E_{av}} - 1}$ (1)

Other useful mathematical results also proven in section **3.** are:

Characterization 1: $E(t) = E_0 e^{\nu t}$ if and only if $\Delta E = \eta\nu$ (2)

Theorem 2a: For any integrable function *E(t)*, $\lim_{t \to s} \dfrac{\eta\nu}{e^{\eta\nu/E_{av}} - 1} = E(s)$ (3)

2. 'Planck-like' Characterization

Let $\eta = \int_s^t E(u)du$ be the 'accumulation of *E*' over a time pulse $\tau = t - s$. We can define $\mathcal{T} = \left(\frac{1}{\kappa}\right)\frac{\eta}{\tau}$ where κ is a scalar constant. The quantity $\mathcal{T}$ behaves like 'temperature'. The faster the accumulation of *E* the higher the $\mathcal{T}$.

Note also that, $E_{av} = \kappa\mathcal{T}$ (4)

By letting *s=0* and using (4) above, we can rewrite (1) as

$$E(t) = E_0 e^{\nu t} \text{ if and only if } E_0 = \frac{\eta\nu}{e^{\eta\nu/\kappa\mathcal{T}} - 1} \quad (5)$$

[Planck's Law](#) for blackbody radiation states that,

$$E_0 = \frac{h\nu}{e^{h\nu/kT} - 1} \tag{6}$$

where E_0 is the energy of radiation, ν is the frequency of radiation and T is the (Kelvin) temperature of radiation (the blackbody), while h is Planck's constant and k is Boltzmann's constant.

Clearly $E_0 = \frac{\eta\nu}{e^{\eta\nu/\kappa\tau} - 1}$ and $E_0 = \frac{h\nu}{e^{h\nu/kT} - 1}$ have the exact same mathematical form, including the 'form' of the quantities that appear in each of these expressions. We can state the main result of this note as,

Result I: *A 'Planck-like' characterization of simple exponential functions*

$$E(t) = E_0 e^{\nu t} \text{ if and only if } E_0 = \frac{\eta\nu}{e^{\eta\nu/\kappa\tau} - 1}$$

Using (3) above we can drop the condition that $E(t) = E_0 e^{\nu t}$ and get,

Result II: *A 'Planck-like' limit of any integrable function*

For any integrable function *E(t)*, $\lim\limits_{t \to 0} \frac{\eta\nu}{e^{\eta\nu/\kappa\tau} - 1} = E_0$

3. Mathematical Derivations *(proofs)*

Notation. We will consistently use the following notation throughout this section of the paper:

$\Delta t = t - s$ is an 'interval of t'

$\Delta E = E(t) - E(s)$ is the 'change of E'

$P = \int_s^t E(u)du$ is the 'accumulation of E'

$\bar{E} = E_{av} = \frac{1}{t-s}\int_s^t E(u)du$ is the 'average of E'

D_x indicates 'differentiation with respect to x'

r is a constant, often an 'exponential rate of growth'

E(t) is any integrable or possibly differentiable function of *t*

Although all the following mathematical derivations make no assumptions as to the variables *t* and *E*, these could be considered to be 'time' and 'energy'. Though many of the proofs given below are very simple, they are included primarily for rigorous consistency and completion.

Part I: exponential functions

We will use the following characterization of exponential functions without proof:

Basic Characterization: $E(t) = E_0 e^{rt}$ *if and only if* $D_t E = rE$

Characterization 1: $E(t) = E_0 e^{rt}$ *if and only if* $\Delta E = Pr$

Proof:

Assume that $E(t) = E_0 e^{rt}$. We have that $\Delta E = E(t) - E(s) = E_0 e^{rt} - E_0 e^{rs}$,

while $P = \int_s^t E_0 e^{ru} du = \frac{1}{r}\left[E_0 e^{rt} - E_0 e^{rs} \right] = \frac{\Delta E}{r}$. Therefore $\Delta E = Pr$.

Assume next that $\Delta E = Pr$. Differentiating with respect to *t*, $D_t E = r D_t P = rE$.

Therefore by the *Basic Characterization,* $E(t) = E_0 e^{rt}$.

q.e.d

Theorem 1: $E(t) = E_0 e^{rt}$ *if and only if* $\frac{Pr}{e^{r\Delta t} - 1}$ *is invariant with respect to t*

Proof:

Assume that $E(t) = E_0 e^{rt}$. Then we have, for *fixed s*,

$$P = \int_s^t E_0 e^{ru} du = \frac{E_0}{r}\left[e^{rt} - e^{rs} \right] = \frac{E_0 e^{rs}}{r}\left[e^{r(t-s)} - 1 \right] = \frac{E(s)}{r}\left(e^{r(t-s)} - 1 \right)$$

and from this we get that $\frac{Pr}{e^{r\Delta t} - 1} = E(s) = \text{ constant}$.

Assume next that $\frac{Pr}{e^{r\Delta t} - 1} = C$ is constant with respect to *t,* for *fixed s*.

Therefore, $$D_t\left[\frac{Pr}{e^{r\Delta t} - 1} \right] = \frac{rE(t)\cdot\left[e^{r\Delta t} - 1 \right] - rP\cdot\left[re^{r\Delta t} \right]}{\left(e^{r\Delta t} - 1 \right)^2} = 0$$

and so, $E(t) = \left(\frac{Pr}{e^{r\Delta t} - 1} \right) e^{r\Delta t} = C \cdot e^{r\Delta t}$ where C is constant.

Letting *t=s* we get *E(s)=C.* We can rewrite this as $E(t) = E(s) e^{r(t-s)} = E_0 e^{rt}$. *q.e.d*

From the above, we have

Characterization 2: $E(t) = E_0 e^{rt}$ *if and only if* $\frac{Pr}{e^{r(t-s)} - 1} = E(s)$

Clearly by definition of E_{av} , $r\Delta t = \dfrac{Pr}{E_{av}}$. We can write $\dfrac{Pr}{e^{r\Delta t} - 1}$ equivalently as $\dfrac{Pr}{e^{Pr/E_{av}} - 1}$ in the above. *Theorem 1* above can therefore be restated as,

Theorem 1a: $E(t) = E_0 e^{rt}$ *if and only if* $\dfrac{Pr}{e^{Pr/E_{av}} - 1}$ *is invariant with respect to t*

The above *Characterization 2* can then be restated as

Characterization 2a: $E(t) = E_0 e^{rt}$ *if and only if* $\dfrac{Pr}{e^{Pr/E_{av}} - 1} = E(s)$

But if $\dfrac{Pr}{e^{Pr/E_{av}} - 1} = E(s)$, then by *Characterization 2a* , $E(t) = E_0 e^{rt}$. So by *Characterization 1,* we must have that $\Delta E = Pr$. And so we can write equivalently $\dfrac{\Delta E}{e^{\Delta E/E_{av}} - 1} = E(s)$. We have the following equivalence,

Characterization 3: $E(t) = E_0 e^{rt}$ *if and only if* $\dfrac{\Delta E}{e^{\Delta E/E_{av}} - 1} = E(s)$

As we've seen above, it is always true that $\dfrac{Pr}{E_{av}} = r\Delta t$. But for exponential functions *E(t)* we also have that $\Delta E = Pr$. So, for exponential functions we have the following result.

Characterization 4: $E(t) = E_0 e^{rt}$ *if and only if* $\dfrac{\Delta E}{E_{av}} = r\Delta t$

Part II: integrable functions

We next consider that *E(t)* is any integrable function. In this case, we have the following.

Theorem 2: i) For any integrable function E(t), $\lim_{t\to s} \dfrac{Pr}{e^{r\Delta t} - 1} = E(s)$

ii) For any differentiable function E(t), $\lim_{t\to s} \dfrac{\Delta E}{e^{\Delta E/E_{av}} - 1} = E(s)$

Proof:

Since $\dfrac{\Delta E}{e^{\Delta E/E_{av}} - 1} \to \dfrac{0}{0}$ and $\dfrac{Pr}{e^{r\Delta t} - 1} \to \dfrac{0}{0}$ as $t \to s$, we apply L'Hopital's Rule.

i) Clearly we have $\lim_{t\to s} \dfrac{Pr}{e^{r\Delta t} - 1} = \lim_{t\to s} \dfrac{E(s)r}{e^{r\Delta t} \cdot r} = E(s)$

ii) Since we are assuming next that *E(t)* is differentiable

$$\lim_{t\to s}\frac{\Delta E}{e^{\Delta E/\bar{E}}-1}=\lim_{t\to s}\frac{D_tE(t)}{e^{\Delta E/\bar{E}}\cdot\left[\dfrac{D_tE(t)\cdot\bar{E}-D_t\bar{E}\cdot\Delta E}{\bar{E}^2}\right]}=\lim_{t\to s}\frac{\bar{E}^2\cdot D_tE(t)}{e^{\Delta E/\bar{E}}\cdot\left[D_tE(t)\cdot\bar{E}-D_t\bar{E}\cdot\Delta E\right]}=E(s)$$

since $\Delta E\to 0$ and $\bar{E}\to E(s)$ as $t\to s$. *q.e.d.*

Corollary A: $\dfrac{\Delta E}{e^{\Delta E/\bar{E}}-1}$ *is invariant with respect to t if and only if* $E(s)=\dfrac{\Delta E}{e^{\Delta E/\bar{E}}-1}$

Proof:

Using *Theorem 2* we have $\lim_{t\to s}\dfrac{\Delta E}{e^{\Delta E/E_{av}}-1}=E(s)$.

Since $\dfrac{\Delta E}{e^{\Delta E/E_{av}}-1}$ is constant with respect to *t*, we have $E(s)=\dfrac{\Delta E}{e^{\Delta E/E_{av}}-1}$.

Conversely, if $E(s)=\dfrac{\Delta E}{e^{\Delta E/E_{av}}-1}$, then by *Characterization 3*, $E(s)=E_0e^{rs}$.

Since $E(s)$ is a constant, $\dfrac{\Delta E}{e^{\Delta E/E_{av}}-1}$ is invariant with respect to *t*. *q.e.d*

Since it is always true by definitions that $r\Delta t=\dfrac{Pr}{E_{av}}$, *Theorem 2* can also be written as,

Theorem 2a: For any integrable function E(t), $\lim_{t\to s}\dfrac{Pr}{e^{Pr/E_{av}}-1}=E(s)$

As a direct consequence of the above, we have the following interesting and important conclusion:

Corollary B: $E(s)=\dfrac{\Delta E}{e^{\Delta E/E_{av}}-1}$ *and* $E(s)=\dfrac{Pr}{e^{Pr/E_{av}}-1}$ *are independent of* Δt *and* ΔE.

Lastly, we state the following simple **mathematical identity:** *(without proof)*

For any integrable function $E(t)$, $\eta=\int_0^{\eta/E_{av}}E(u)du$, *where* $\eta=\int_0^{\tau}E(u)du$ *and* $E_{av}=\dfrac{\eta}{\tau}$

4. Appendix

In this appendix we provide a direct and independent proof of *Characterization 3* and include some other interesting connections and further discussions.

We first prove the following,

Lemma: For any *E*, $D_t\bar{E}(t)=\dfrac{E(t)-\bar{E}}{t-s}$ and $D_s\bar{E}(s)=\dfrac{\bar{E}-E(s)}{t-s}$

Proof:

We let $\Delta t = t - s$ and $\bar{E} = \frac{1}{t-s}\int_s^t E(u)du$.

Differentiating with respect to *t* we have $(t-s)\cdot D_t\bar{E}(t) + \bar{E} = E(t)$

Rewriting, we have $D_t\bar{E}(t) = \frac{E(t) - \bar{E}}{t-s}$.

Differentiating with respect to *s* we have $(t-s)\cdot D_s\bar{E}(s) - \bar{E} = -E(s)$

Rewriting, we have $D_s\bar{E}(s) = \frac{\bar{E} - E(s)}{t-s}$.

q.e.d.

Characterization 3: $E(t) = E_0 e^{rt}$ *if and only if* $\frac{\Delta E}{e^{\Delta E/E_{av}} - 1} = E(s)$

Proof:

Assume that $E(t) = E_0 e^{rt}$. From,

$$P = \int_s^t E_0 e^{ru}du = \frac{E_0}{r}\left[e^{rt} - e^{rs}\right] = \frac{E_0 e^{rs}}{r}\left[e^{r\Delta t} - 1\right] = \frac{E(s)}{r}\left[e^{r\Delta t} - 1\right]$$

we get, $E(s) = \frac{Pr}{e^{r\Delta t} - 1}$. This can be rewritten as, $E(s) = \frac{Pr}{e^{Pr/E_{av}} - 1}$.

Since $\Delta E = Pr$, this can further be written as $E(s) = \frac{\Delta E}{e^{\Delta E/E_{av}} - 1}$.

Conversely, consider next a function $E(s)$ satisfying

$$E(s) = \frac{\Delta E}{e^{\xi} - 1}, \quad \text{where} \begin{cases} \Delta E = E(t) - E(s) \\ \Delta t = t - s \\ \xi = \frac{\Delta E}{\bar{E}} \\ \bar{E} = \frac{1}{\Delta t}\int_s^t E(u)du \end{cases} \quad \text{and } t \text{ can be any real value.}$$

From the above, we have that $e^{\xi} = \frac{\Delta E}{E(s)} + 1 = \frac{E(t) - E(s) + E(s)}{E(s)} = \frac{E(t)}{E(s)}$.

Differentiating with respect to *s*, we get $e^{\xi}\cdot D_s\xi = \frac{-E(t)\cdot D_sE(s)}{E(s)^2} = -e^{\xi}\cdot\frac{D_sE(s)}{E(s)}$

and so, $$D_s\xi = -\frac{D_sE(s)}{E(s)} \qquad (1)$$

From the above *Lemma* we have that $$D_s\bar{E}(s) = \frac{\bar{E} - E(s)}{t-s} \qquad (2)$$

Differentiating $\xi = \frac{\Delta E}{\bar{E}}$ with respect to s we get,

$$D_s\xi = \frac{-D_sE(s)\cdot\bar{E} - \Delta E\cdot D_s\bar{E}(s)}{\bar{E}^2} \qquad (3)$$

and combining (1), (2), and (3) we have

$$-\frac{D_sE(s)}{E(s)} = \frac{-D_sE(s)\cdot\bar{E} - \frac{\Delta E}{\Delta t}\left(\bar{E} - E(s)\right)}{\bar{E}^2} = -\frac{D_sE(s)}{\bar{E}} - \frac{\Delta E}{\Delta t}\cdot\frac{\left(\bar{E} - E(s)\right)}{\bar{E}^2}$$

We can rewrite the above as follows,

$$\frac{D_sE(s)}{E(s)} - \frac{D_sE(s)}{\bar{E}} = D_sE(s)\left(\frac{\bar{E} - E(s)}{E(s)\cdot\bar{E}}\right) = \frac{\Delta E}{\Delta t}\cdot\frac{\left(\bar{E} - E(s)\right)}{\bar{E}^2}$$

and so, $\frac{D_sE(s)}{E(s)} = \frac{\Delta E}{\Delta t}\cdot\frac{1}{\bar{E}}$.

Using (1), this can be written as $-D_s\xi = \frac{\xi}{\Delta t}$, or as $\xi = -D_s\xi\cdot\Delta t$. (4)

Differentiating (4) above with respect to s, we get $D_s\xi = -D_s^2\xi\cdot\Delta t + D_s\xi$.

Therefore, $D_s^2\xi = 0$. Working backward, this gives $D_s\xi = -r = \text{constant}$.

From (1), we then have that $\frac{D_sE(s)}{E(s)} = r$ and therefore $E(s) = E_0e^{rs}$. *q.e.d.*

Further Discussion:

The formula $E(s) = \frac{\Delta E}{e^{\Delta E/E_{av}} - 1}$ can be interpreted as saying that the 'instantaneous value' of the quantity E can be calculated exactly if we knew the 'change' and the 'average' of E over some time interval. Thus if we knew the value of ΔE and E_{av}, by substituting these values in this formula we can calculate the exact 'instantaneous value' of E. Furthermore, we would get the same value of E regardless of the interval of time over which the values of ΔE and E_{av} were taken. That is to say, the formula is independent of Δt *(Corollary B)*.

Consider a 'black box' containing some quantity E. Although we may not be able to measure the exact (absolute) 'instantaneous' value of E directly, if we have instruments that can measure the 'change of E' and the 'average of E' over some time interval, and if the specific interval is not relevant (as it shouldn't be if there is just one exact value of *E* in the box at any one instant), then using this formula we could calculate the exact (absolute) 'instantaneous' value of E. In a sense, the instrument 'samples' the box by measuring ΔE and E_{av}. From these values we can then calculate from the formula the exact 'instantaneous' value of E in the box.

Note further that for any function *E(t)*, the expression $\frac{Pr}{e^{r\Delta t} - 1}$ can also be written as $\frac{\int_s^t E(u)du}{\int_0^{t-s} e^{ru}du}$.

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Other related papers by the author:

1) "Planck's Law is an Exact Mathematical Identity"
2) "A Simple Stock Comparison Model and Planck's Law in Quantum Physics"
3) "The Temperature of Radiation"
4) "A Plausible Explanation of the Double-slit Experiment in Quantum Physics"

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